Pattern Avoidance in Permutations

A B. Tech Project Report Submitted
in Partial Fulfillment of the Requirements
for the Degree of

Bachelor of Technology

by

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CERTIFICATE

*This is to certify that the work contained in this thesis entitled "**Pattern Avoidance in Permutations**" is a bonafide work of **Abhilasha Sancheti (Roll No.130101083)** and **Kunal Jain (Roll No.130101042)**, carried out in the Department of Computer Science and Engineering, Indian Institute of Technology Guwahati under my supervision and that it has not been submitted elsewhere for a degree.*

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April, 2018

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Acknowledgements

We would like to extend our sincere gratitude towards Prof. Benny George for guiding and constantly encouraging us during the course of this project.

Contents

Abstract

Permutations have a vast variety of combinatorial structure. Permutation of a finite set can be represented in many equivalent ways such as a collection of cycles, function, matrix, etc. Each of these representations is outcome of some transformations or operations applied on a permutation.

These operations or transformations on permutations include descent set, excedance set, cycle type, subsequences, composition (product), etc. The main aim of the project is to look at the combinatorial aspects of pattern avoidance which is an active area of research.

Chapter 1

Introduction

1.1 Organization of The Report

This chapter provides a background for the topics covered in this report. This report is dedicated to the study of permutations, its properties and combinatorial aspect associated with it. We start with the definition of a permutation and notations that we are going to follow in the rest of the report. In Chapter 2 we describe the literature review that we have done. We start with permutations as a Linear Orders. Then describe some properties related to it. In the next section of the chapter we introduce the notion of pattern avoidance and some findings related to this area. In Chapter 3 we introduce some classes of problems in the area of pattern avoidance and then define the problems that we are going to work. In Chapter 4 we will establish the problem statement by delving into the sub-issues related to pattern avoidance and finally in Chapter 5 we discuss the work we did and the results that we derived during the course of the project.

1.2 Definitions

Definition 1.2.1 *A* permutation *is defined as a linear ordering of the elements of the set [n]= 1, 2, 3, 4, , n. If it consists of n entries then it is also called an n-permutation.*

There are n! permutations possible for an n-permutation. This is because, consider $p = p_1p_2p_3....p_n$ we have n choices for p_1, n-1 for p_2 and similarly 1 for p_n thus a total of n!.

Definition 1.2.2 *For a permutation $p = p_1p_2p_3....p_n$, we say that a permutation p^r is* reverse *of p if $p^r = p_np_{n-1}...p_1$*

Definition 1.2.3 *For a permutation $p = p_1p_2p_3....p_n$, we say that a permutation p^c is* complement *of p if i^{th} entry of p^c is equal to $n + 1 - p_i$.*

Definition 1.2.4 *Consider a sequence of positive real numbers as $a_1, a_2, ..., a_n$. If an index i exists such that $1 \leq i \leq n$, and $a_1 \leq a_2 \leq ...... \leq a_i \geq a_{i+1}..... \geq a_n$ then we say that the sequence is* unimodal.

Definition 1.2.5 *We say that the sequence of positive real numbers $a_1, a_2, ..., a_n$. is* log-concave *if $a_{i-1}a_{i+1} \leq a_i^2$ is satisfied for all indices i.*

If a sequence of positive real numbers is log-concave then it is also unimodal.

Definition 1.2.6 *Consider $p = p_1p_2p_n$ to be a permutation. If $p_i > i$ then we say that i is an* excedance *of p.*

Definition 1.2.7 *Consider two sequences $\phi = [a_1, a_2, .., a_m]$ and $\chi = [b_1, b_2, ..., b_m]$ of equal length m. They are said to be* order isomorphic *if, $\forall i, j \quad a_i < a_j$ if and only if $b_i < b_j$*

Formally, we say that ϕ is order isomorphic to χ if and only if $\forall i, j \quad a_i < a_j \Leftrightarrow b_i < b_j$
Result: Every sequence ϕ is order isomorphic to a unique permutation of length m of 1,2,...,m.
Throughout our study we focus our work on permutations of [n] as can be seen from the result above, each sequence maps uniquely to these permutations.

Definition 1.2.8 *Consider two sequences* $\phi = [a_1, a_2, .., a_m]$ *and* $\chi = [b_1, b_2, ..., b_n]$ *of length* m *and* n *respectively. We say* ϕ *is involved in* χ*, if there exists a subsequence of* χ *which is order isomorphic to* ϕ*. We denote* sequence involvement *with* $\preceq$.

Example: $\Phi = [3, 1, 4, 2, 5]$ $\chi = [7, 2, 1, 5, 8, 3, 6, 9, 4]$ $\Phi \preceq \chi$ as subsequence [7, 5, 8, 6, 9] of χ is order isomorphic to Φ.

Definition 1.2.9 *It is* NP complete *to decide that for given two permutations* ϕ *and* χ*, if they are involved.*

Chapter 2

Review of Prior Works

This chapter discusses about permutations and its combinatorial aspects that we have been studying. We describe below some properties of permutations, enumerate specific type of patterns and present the consequent results.

2.1 Permutations as Linear Orders

Increasing sequence is the most ordered one as an entry is always followed by an entry greater than it. We are interested in the disorders in the permutations such as when an entry in permutation is followed subsequently by an entry which is smaller in ordered criterion. [Bón04]

2.1.1 Descents

Definition 2.1.1 *Consider a permutation $p = p_1p_2p_3....p_n$. If $p_i > p_{i+1}$, then we say that i is the* descent *of p and if $pi < p_{i+1}$, then we say that i is the* ascent *of p.*

The set which contains all the descents of a permutation p is called the *descent set of p*. We denote it as D(p). We are interested in combinatorics related to this property. The following are some results:

1. Number of permutations whose descent set is contained in A.

   Consider A=$a_1, a_2, a_3 ... a_k \subseteq [n-1]$ and $\alpha(A)$ to be the number of n-permutations whose descent set is contained in A, then

$$\alpha(A) = \binom{n}{a_1}\binom{n-a_1}{a_2-a_1}\binom{n-a_2}{a_3-a_2}......\binom{n-a_k}{n-a_k}$$

2. Number of permutations with a descent set A.[Bón11]

   Consider $A \subseteq [n-1]$. We denote the number of permutations with descent set A by $\beta(A)$. Thus by using the inclusion and exclusion principle we can say that

$$\beta(A) = \sum_{T \subseteq A} (-1)^{|A-T|} \alpha(A)$$

3. Number of permutations with a given number of descents.

   We define S(n,k) as the number of n-permutations with k-1 descents. S(n,k) is called the Eulerian number. The explicit formula for Eulerian numbers, for all non-negative integers n,k which satisfies the condition that $k \leq n$, is given by:

$$S(n,k) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n$$

   Eulerian numbers follow a recurrence relation. The recurrence formula for all positive integers k and n which satisfies the condition that $k \leq n$ is given by:

$$S(n,k+1) = (k+1)S(n-1,k+1) + (n-k)S(n-1,k)$$

   Properties of Eulerian numbers:

   For any positive integer n S(n,k) is log-concave.

   For any fixed n, S(n,k) has real roots only.

   The number of n-permutations with k-1 excedances is S(n, k).

2.1.2 Runs

Definition 2.1.2 *Consider $p = p_1p_2......p_n$ to be a permutation p. By definition of descents, between each descent there is an increasing sequence of consecutive elements. Thus if sequence has d - 1 descents, it will have d increasing subsequences. We call them as the* ascending runs *of p.*

The notion of ascending runs cans be extended to alternating runs.

Definition 2.1.3 *Consider $p = p_1p_2......p_n$ to be a permutation.If either $p_{i-1} < p_i > p_{i+1}$, or $p_{i-1} > p_i < p_{i+1}$ then p is said to change direction at i. Alternatively p changes direction if there is a peak or a valley.*

The number of n-permutations with k *alternating runs* is denoted as H(n,k). As Eulerian numbers S(n,k), H(n,k) also has real roots only and follows a similar recurrence relation given below: Consider H(1, 0)=1, and H(1, k)=0 for $k > 0$. For positive integers n and k

$$H(n, k) = kH(n-1, k) + 2H(n-1, k-1) + (n-k)H(n-1, k-2)$$

2.1.3 Inversions

In the last section we discussed descents of permutations where an entry was greater than the entry directly following it. Now we will look at inversions where an entry is followed by some entry smaller than it.

Definition 2.1.4 *Consider a permutation $p = p_1p_2......p_n$.We say that (p_i, p_j) is an* inversion *of p if $i < j$ but $p_i > p_j$.*

We conventionally denote the number of inversions of p by i(p) and the size of set of n sized permutations which have k inversions by c(n,k). For a n-permutation i(p) can be at most $\binom{n}{2}$. For any fixed size permutation n, the sequence c(n,0),c(n,1),..........c(n,$\binom{n}{2}$) is log-concave.

Similar to the previous two numbers S(n,k) and H(n,k), c(n,k) also follows a recurrence relation given by: For $n \geq k$,

$$c(n+1, k) = c(n+1, k-1) + c(n, k).$$

Definition 2.1.5 *Consider a permutation $p = p_1 p_2......p_n$. The sum of the descents of p is called as the* major index *or greater major index. We denote it as maj(p). That is,*

$$maj(p) = \sum_{i \varepsilon D(p)} i$$

Result: For all non-negative integers k and all positive integers n, the size of set of n sized permutations with k inversions is same as the size of set of n sized permutations with major index k.

2.2 Pattern Avoidance

In the previous section we discussed about inversions where a pair of entries of the permutation were related to each other in some manner and can be present anywhere in it. The more general notion of this relationship between pair of entries could be extended to k-tuples of entries [Bón04].

We say a pattern q is present in a permutation p(longer or equal to length of q) if any subsequence of p of length same as that of q can be reduced to q. Consider a permutation p=241563, and q=213. We can then say that 3-tuple (4,1,5) of p forms a pattern of type (2,1,3) because (4,1,5) can be reduced to (2,1,3) by mapping 4 to 2, 1 to 1 and 5 to 3 respectively.

We say that p avoids a pattern q if none of the subsequence of p can be reduced to q. Formally, we can define pattern avoidance as follows:

Definition 2.2.1 *Consider* $q = (q_1, q_2, .., q_k)\,\varepsilon A_k$ *to be a permutation, and let* $k \leq n$. *We say that the* permutation $p = (p_1, p_2, .., p_n)\,\varepsilon A_n$ *is* q-avoiding *if and only if there is no* $1 \leq i_{q_1}, < i_{q_2}, < .... < i_{q_k} \leq n$ *such that* $p(i_1) < p(i_2) < ... < p(i_k)$.

Alternatively, we can say that A_n is A_q avoiding if there exists no subsequence of A_n which is order isomorphic to A_q. Clearly if A_n avoids A_q,A_n and A_q are not involved.

Example: Consider A_3 = [1, 3, 2] and A_5= [1, 2, 3, 4 ,5]

A_5 is A_2 avoiding as there exists no subsequence of A_5 which is order isomorphic to A_2. So far we have counted number of n-permutations with a given number of inversions. Now we are interested in finding $A_n(q)$ that, is number of q pattern avoiding n-permutations.

We have $A_n(12) = A_n(21) = 1$, so the first case that is non-trivial and need attention is of patterns of length three. There are in all 3! possible permutations of length three. So we have six such patterns, but there are many symmetries between them which we will see further. Recall that for a permutation $p = p_1p_2p_3....p_n$, the reverse of p is defined as the permutation $p^r = p_np_{n-1}p_{n-2}....p_1$, and the p permutations' complement p^c whose i^{th} entry is $n+1-p_i$ It can be seen very easily that 123 avoiding permutations' reverse avoids 321 pattern, thus $A_n(123) = A_n(321)$. Similarly, 132 avoiding permutations' reverse avoids 231, its complement avoids 312, and the complements' reverse avoids 213. Similarly we also have $A_n(132) = A_n(231) = A_n(312) = A_n(213)$. It can be shown that $A_n(123) = A_n(132)$ [SS85] and thus all the length three permutations are avoided by same count of n sized permutations. We will also enumerate the permutations avoiding the combinations of patterns of length three. The following theorem by will be helpful in the enumeration.

Theorem: 1 *If there is a sequence of* $n^2 + 1$ *numbers, then there is either a monotonically increasing subsequence of n+1 numbers or a monotonic decreasing subsequence of n+1 numbers.*

There are a total of $2^6 = 64$ possible combinations including the empty set. We denote

T as the combination to be avoided. We will discuss the enumeration of permutations avoiding some of these combination below:

1. T = {123}. The decreasing permutation is the only one that avoids 123 because in all the other permutations we can find i,j and k such that $p_i < p_j < p_k$. Thus $A_n(T) = 1$. As we have previously shown that the number of permutations avoiding the singleton sets is same. So, $A_n(\{123\}) = A_n(\{132\}) = A_n(\{213\}) = A_n(\{231\}) = A_n(\{312\}) = A_n(\{321\}) = 1$.

2. T = {123,132,213,231,312,321}. By Theorem 1, it is obvious that for permutations of length greater than nine $A_n(T) = 0$.

3. T = {123,132,213,231,312}. The only permutation avoiding this set is the decreasing sequence. So $A_n(T) = 1$.

4. T = {123,213,231,312,321}.By the same argument as in case 2 $A_n(T) = 0$. Similarly for T={123,132,231,312,321}, T= {123,132,231,312,321}, T = {123,132,213,312,321}, T={123,132,213,231,321},T ={132,213,231,312,321} , $A_n(T) = 0$.

5. T = {123,132 213,231}. The only patterns allowed in the permutation are 312 and 321 so the for each triplet the first entry has to be maximum of the rest two thus permutation p should be such that $p_1 > p_2 > p_3 > ..... > p_{n-2}$ and last two positions could be anything. Thus there are only two possible permutations. So, $A_n(T) = 2$.

6. T = {123,132 213,312}. The allowed patterns are 231 and 321. Thus similar to the above reason p should be such that $p_n < p_{n-1} < p_{n-2} < ..... < p_3$ and first two entries could be any of the largest two. So $A_n(T) = 2$.

7. T = {123,132, 231,312}.The allowed patterns are 213 and 321. Let us consider four elements A,B,C and D such that $A < B < C < D$. Now look at the possibilities at each position a_{i+t} for t=0,1,2,3. We observe that only D and C can come at a_i, C

and B at a_{i+1}, A and B at a_{i+2}, and D and A at a_{i+3}. Also see that if element at any one of the position is fixed then rest all are also fixed. Thus we have $A_n(T) = 2$.

8. T = {123,213,231,312}.With the same argument as above we observe that only A and D can come at a_i, C and D at a_{i+1}, B and C at a_{i+2}, and A and B at a_{i+3}. Also see that if element at any one of the position is fixed then rest all are also fixed. Thus we have $A_n(T) = 2$.

9. T={132,213,231,312}. The only allowed patterns are 123 and 321. So the possible permutations are increasing and decreasing sequences. Thus $A_n(T) = 2$.

10. For T= {132,213,231,321},T={132,213,231,321} and T= {213,231,312,321}, $A_n(T) = 2$ because of the similar argument as in the above cases.

11. Similarly for combinations of two and three the results are mentioned in [SS85]

A general formula for the number of three length pattern avoiding permutations $A_n(q)$ is given by Catalan numbers C_n

$$A_n(q) = C_n = \frac{\binom{2n}{n}}{n+1}$$

2.3 Conclusion

This chapter provided details of some of the properties related to permutations. We also enumerated the permutations with given number of descents, alternating runs and inversions. It also presents some of the existing works in the field of pattern avoidance. The exact enumeration for the avoidance of three length patterns avoiding permutations were also discussed.

Chapter 3

Problem Classes and Future Work

The pattern avoidance of length 3 in permutations is a much studied topic. However there are not many results for pattern avoidance of length greater than three and pattern avoidance in compositions. In this chapter we define some of the open problems in pattern avoidance, briefly discuss them and describe our proposed future work.

3.1 Problems

3.1.1 Pattern Avoidance of one pattern of Length 4

There are 4! permutations of P = [1,2,3,4].

Table 3.1 Classes avoiding one pattern of length 4

Pattern	Enumeration Sequence	Type	Exact Enumeration
[1 , 3, 4, 2] [2, 4, 1, 3]	1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, 3475090, ...	Not RationalAlgebraic	Miklos Bona(1997) [Bón97]
[1, 2, 3, 4] [1, 2, 4, 3] [1, 4, 3, 2] [2, 1, 4, 3]	1, 1, 2, 6, 23, 103, 513, 2761, 15767, 94359, 586590, 3763290, 24792705, 167078577, ...	Holonomic	Ira M. Gessel(1990) [Ges90]
[1, 3, 2, 4]	1, 1, 2, 6, 23, 103, 513, 2762, 15793, 94776, 591950, ...	No Known Form	No exact enumeration

There is no exact enumeration of pattern avoidance of class 1324-avoiding permutations.

Marinov, RadoiiRado in their paper published in 2013 [MR03] provide a recursive formula, which was upper bounded by Bona in 2015 [Bón15] and lower bounded by Bevan in 2015 [Bev15].

A tighter bound and closed formula is an open research topic.

3.1.2 Pattern Avoidance of two patterns of Length Four

There constitute 56 symmetry classes out of which 8 remain to be enumerated.

Table 3.2 Classes avoiding two pattern of length 4

Permutation	Results
[4, 2, 3, 1] [4, 1, 2, 3]	Conjectured to not satisfy any closed form
[4, 1, 2, 3] [3, 4, 1, 2]	No Results
[4, 3, 1, 2] [4, 1, 2 ,3]	Conjectured to not satisfy any closed form
[3, 4, 1, 2] [2, 4, 1, 3]	No Results
[4, 3, 2, 1] [4, 2, 3, 1]	Conjectured to not satisfy any closed form

Lower and Upper bound and an exact formula recursive or non recursive is an open research topic.

3.1.3 Classes which contain some pattern and avoid some pattern

We are interested in those classes which avoid some pattern with an extra condition that they must contain some pattern.

For example: We are interested in studying in all 1234 permutations which avoid 1324.

This topic is very sparsely studied and answers many interesting questions like all patterns which contain increasing sub-sequences of 4, which doesn't contain any decreasing sub sequence of length 4.

The results of this class is a strict subset of class that avoid pattern, and there relation with those classes will be an interesting topic.

3.1.4 Classes with permutations with restricted patterns

We are interested in those classes of permutations which contains some patterns but restricted number of times.

For example: We are interested in studying all permutations which contain 213 pattern exactly r times.

A more restricted form of this class (subarray with 213 pattern) forms a interesting questions in high school mathematics. Krattenthaler published an interesting paper in 2000 [Kra01] in which he established a precise asymptotic estimate in number of permutations which avoid 1,3,2 pattern with contain d occurrences of the pattern 1,2,3,...,k.

3.1.5 Classes which avoid composite patterns

We are interested in those classes of permutations which avoid 2 or more patterns simultaneously. These patterns can be of same length or varying length.

For example: We are interested in studying in all permutation which avoid 1234 and 132 patterns. The result of this is an interesting result of alternate Fibonacci numbers. That is for n size permutations answer is F(2n - 1) where F(n) is defined to be the nth Fibonacci number.

For two pattern avoidance with either length 3 and length 4 or both, is a much studied topic and all the results are enumerated for two pattern avoidance of length 3, one pattern of length 3 and one pattern of length 4, and as discussed in above class 8 classes of two pattern avoidance of length 4 remain to be enumerated.

However composite pattern avoidance with more than two patterns is very sparsely studied and the results are very interesting for the few classes which are studied.

3.2 Conclusion

We are interested in working on these 5 classes of problem that are described above.

1. The missing exact numeration of 1324 pattern avoidance is a much studied problem, and our interest is to tighter bound the current result.

2. The second class of problem has two classes for which there are no results in terms of bounds or recursive structure. Our interest is to get a bound on these results.

3. The third and fourth class of problems can be generalized to a more common class of permutations which avoid a certain function.

4. The fifth class of problem is our main interest. There are some interesting analogies of these problems with the existing mathematical problems. We will start our work with this problem, mainly looking into avoid 3 or more composite patterns of length 3 and 4.

Chapter 4

Problem Establishment

In Chapter 2 we introduced the notion of Pattern Avoidance in Permutations and in Chapter 3 we discussed the various problem classes which we identified and are interested in working. In this Chapter we will look at the more specified types of pattern avoidance problems and the sub-issues involved in its study.

4.1 Sub-issues Involved in Pattern Avoidance

Let T be any set of permutations. We say that a permutation p *avoids* T if it avoids all patterns $t \in T$. We are interested in counting the number of permutations avoiding T. The general problem is difficult because of the following algorithmic issues involved with it.

1. Given a set T, whether there exists at least one permutation avoiding it?.

   There is no permutation which avoids all the patterns of length 2.

2. What is the length of the smallest permutation avoiding T if it exists?

   The answer to this question depends upon the length of the patterns contained in T. Permutations of length lesser than that of the pattern trivially avoid that pattern.

3. How does the number of n-permutations avoiding T varies with n?

   Catalan number enumerates the permutations avoiding patterns of length 3. The growth of this enumeration is also dependent on the set T.

4. What should be the smallest length of permutations to be looked at in case of non-uniform set T?

   We define pattern avoidance as follows:

   A permutation p is said to avoid patterns in T if none of its subsequence can be reduced to any of the patterns in T provided the length of p is greater than or equal to the largest length pattern.

5. Count the number of permutations which allows some patterns to be present not more than certain number of times?. This is known as restricted pattern avoidance.

6. What is the number of equivalence classes for patterns of length n?.

   Patterns which are avoided by the same number of permutations belong to the same equivalence class. To establish this we know that if a permutation p avoids a pattern q then the reverse permutation p^r avoids pattern q^r and the compliment of the permutation p^c avoids q^c. But apart from these symmetries there might exist other similarities depending upon the pattern itself.

7. Can an analogy similar to *Erdos Szekeres* theorem [ES35] be defined for other pair of patterns?

   Lets us consider any other two patterns except increasing and decreasing, say T={123,321}. The decreasing sequence of any length $n > 3$ avoids T. Similarly if we consider T= {321,213}, then both increasing and decreasing sequences of length $n > 3$ avoids T. Thus similar analogy cannot be derived for any other combination. Hence we concentrate on avoiding monotonic subsequences.

4.2 Conclusion

In this chapter, we discussed the various sub-issues which we identified and worked on. We also discussed the difficulties in studying these sub-issues. Thus we decided to work on relation between length of permutation and the number of monotonically increasing or decreasing subsequences appearing in it.

Chapter 5

Results

In the analysis of the enumeration of permutations avoiding different subsets of patterns of length three, we have used the result of the *Erdos Szekeres* theorem [ES35]. In this Chapter we will look into further extensions to this theorem.

5.1 Extension of Erdos Szekeres Theorem

Theorem: 2 *If there is a sequence of $n^2 + 1$ numbers, then there is either a monotonically increasing subsequence of n+1 numbers or a monotonically decreasing subsequence of n+1 numbers.*

We can refer to "Five Proofs of Subsequence Theorem An Exposition By William Gasarch" for various proofs of this theorem. □

Erdos Szekeres Theorem talks about the presence of a monotonically increasing or decreasing subsequence, we are interested in number of such sequences given permutation length η.

Formally we can define the problem as finding minimum number of distinct monotonic subsequences of length $n + 1$ in a permutation of size η for given parameters n and η.

The possible solutions and bounds which we found for the problem described above are discussed below.

1. $\eta = k(n^2+1)$. It can be seen as an extension to Erdos Szekeres. A sequence of length $k(n^2+1)$ can be divided into k sequences of length (n^2+1). So, according to Erdos Szekeres each of these sequences will contain a monotonic subsequence of length $n+1$. Thus k sequences will contain k monotonic subsequences of length $n+1$.

2. $\eta = (n+k-1)^2+1$. According to the theorem there exists a monotonic subsequence of length $n+k$. This subsequence can be decomposed to get k monotonic subsequences of length $n+1$.

3. $\eta = n^2+k$. A sequence of length n^2+k can be decomposed to get k monotonic subsequences of length n^2+1. We can get k distinct monotonic subsequences in the following way. By Erdos Szekeres in the first n^2+1 elements there exists at least one monotonic subsequence of length $n+1$. We can find remaining k-1 subsequences by removing an element from the subsequence found, and adding one element from the remaining $k-1$ elements and applying the Erdos Szekeres Theorem. We can continue this process exactly k times, each time generating a distinct subsequence since we remove one element each time and add a new one.

It turns out as we discuss later that the bound defined by the last solution is in fact tight. Before proving the tightness, we prove the tightness of Erdos Szekeres theorem.

Theorem: 3 *There exists a sequence of length* n^2+1 *such that it has exactly one monotonic subsequence of length* $n+1$.

Proof. We can construct such a sequence in the following way:

1. Concatenate $(n-1)$ monotonically decreasing subsequences of length n in a way such that the minimum element of the subsequence $i+1$ is greater than the maximum element of the subsequence i.

2. Now append remaining $(n+1)$ elements in the decreasing order to the previously constructed larger sequence such that the minimum of the remaining elements is

greater than the maximum of the last sequence concatenated in the above step.

For example to construct a sequence of length 5 such that $n = 2$, by step 1 of the construction we have $n - 1 = 1$ so only one sequence of length 2 which is $\{21\}$. In step 2 we append $\{543\}$ to sequence obtained in step 1. So the final sequence is $\{21543\}$ and it has only 1 monotonic subsequence of length 3 which is $\{543\}$. The proof of **Theorem** 3 follows from the following two claims.

Claim 1: There does not exist any monotonically increasing subsequence of length $n + 1$ in the sequence constructed above.

Proof. In order to form an increasing subsequence of length $n+1$ we have to take 2 elements from at least one of the n subsequences constructed above, but each of the subsequence is monotonically decreasing so we cannot take more than one element from any of the n subsequences. Thus, at most one element can only be taken from each of the n subsequences, so that a monotonic subsequence of length $n+1$ cannot be formed from the above constructed sequence. □

Claim 2: There exists exactly one monotonically decreasing subsequence of length $n + 1$ in the sequence constructed above.

Proof. Decreasing subsequences cannot be formed by taking elements from different subsequences constructed and there exist only one subsequence of length $n + 1$ which is monotonically decreasing by construction. □

Now we state the solution we discussed above as a theorem, and prove its tightness.

Theorem: 4 *A sequence of length $n^2 + k$ has at least k distinct monotonic subsequences of length $n + 1$ for $k \leq n$.*

Proof. By applying Erdos Szekeres Theorem k times on the sequence as described above, we can get k distinct monotonic subsequences of length $n + 1$. □

Theorem: 5 *There exists a sequence of length n^2+k which has exactly k distinct monotonic subsequences of length $n + 1$ for $k \leq n$.*

We can construct a sequence of length $n^2 + k$ which has exactly k monotonic subsequences of length $n + 1$. The construction is similar to that provided in $Theorem 4$.

1. Concatenate $(n-k)$ monotonically decreasing subsequences of length n in a way such that the minimum element of the subsequence $i + 1$ is greater than the maximum element of the subsequence i.

2. Now append k monotonically decreasing subsequences of length $(n + 1)$ to the previously constructed larger sequence such that the minimum element of each subsequence is greater than the maximum of the previous subsequence.

The bound is said to be tight if we can show that there exists a sequence of length n^2+k-1 which contains less than k monotonic subsequences. This is evident from the construction of sequence provided above with length of the sequence to be $n^2 + k - 1$. □

The above theorem works for cases $k \leq n$. We discuss the case $k > n$.

We provide a construction which has $O(n^\alpha)$ distinct monotonically decreasing subsequences and no monotonically increasing subsequences for a given sequence of length $n^2 + \alpha n$. A very weak theoretical lower bound can be obtained by applying *Erdos Szekeres* theorem on $n^2 + \alpha n$ sequence. This lower bounds to $O(n^{\alpha/2})$. The construction is defined in following manner.

1. Construct n monotonically decreasing subsequence of length $n + \alpha$ which we will call buckets in a way such that the minimum element of the subsequence $i + 1$ is greater than the maximum element of the subsequence i.

There are no distinct monotonically increasing subsequences of length $n + 1$ in our constructed sequence.

Since our construction is such that elements of $i + 1$ bucket is strictly larger than the elements of i bucket. Since there are n such buckets, to construct an increasing subsequence of length $n + 1$ we need to take at least two elements from a single bucket but by our construction elements in a bucket are decreasing, there cannot be any increasing subsequence of length $n + 1$ by taking elements from more than 1 bucket.
Within a bucket, there are $n+\alpha$ decreasing elements. So we cannot construct an increasing subsequence of $n + 1$ elements by taking elements from a single bucket.
There are $O(n^\alpha)$ distinct monotonically decreasing subsequences in our constructed sequence.
Since our construction is such that elements of $i + 1$ bucket is strictly larger than the elements of i bucket so it is not possible to form decreasing subsequence of length $n + 1$ by taking elements from more than two buckets.
Within a bucket, there are $n + \alpha$ decreasing elements. So we can pick $n + 1$ elements from this bucket and they will form a decreasing subsequence. So we get $\binom{n+\alpha}{n+1}$ distinct monotonically decreasing subsequences. We have n such buckets so in total we have

$$n * \binom{n+\alpha}{n+1} \approx O(n^\alpha)$$

Also having both increasing and decreasing subsequences will result in greater number than our proposed construction, and if we allow subsequences to be constructed by picking elements from more than a bucket will result in exponential number of subsequences.

Experimental Evidence: We counted the number of patterns appearing for $n = 3, \alpha = 2$. The size of permutation set will be

$$(n^2 + \alpha * n)! = 15! \approx 1.3076744e + 12$$

and each permutation has $\binom{15}{4} = 1365$ subsequences of length 4. This enormous size makes it computationally not feasible to check.

We randomly generated 1 million permutations and checked for the count of monotonically increasing or decreasing subsequences. We also used local search algorithms on this problem to find the local minimum count. In both the experiments we couldn't find any permutation with less than 15 monotonically increasing or decreasing subsequences which is consistent with our belief since

$$n * \binom{n+\alpha}{n+1} = 3 * \binom{5}{4} = 15$$

is the count which we get from our construction. However due to lack of formal proof we state the above belief as a conjecture.

Conjecture: A sequence of length $n^2+\alpha n$ has $O(n^\alpha)$ distinct monotonic subsequences of length $n+1$.

5.2 Future Work

The conjecture we stated remains to be proven or disproven. We are interested in exploring this problem further and either proving that the bound we found is in fact tight or improving the bound.

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